DEC 5 1960
NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y.

# Institute of Mathematical Sciences

## MAGNETO-FLUID DYNAMICS DIVISION

MF-10 NYO-2886

# ON THE GREEN'S FUNCTION FOR TWO-DIMENSIONAL MAGNETOHYDRODYNAMIC WAVES. I

by

Harold Weitzner

April 14, 1960

AEC Research and Development Report

NEW YORK UNIVERSITY

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, express or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

Magneto-Fluid Dynamics Division
Institute of Mathematical Sciences
New York University

Physics and
Mathematics

MF-10 NYO-2886

# ON THE GREEN'S FUNCTION FOR TWO-DIMENSIONAL MAGNETOHYDRODYNAMIC WAVES. I

by

Harold Weitzner

April 14, 1960

AEC Research and Development Report

Contract No. AT(30-1)-1480

# ABSTRACT

The Green's function is evaluated for the Lundquist equations linearized about uniform magnetic field, density and zero flow velocity. It is further assumed that all quantities vary with two space variables and time only and that the constant magnetic field lies in the plane of the two space variables.

# TABLE OF CONTENTS

Page

Abstract . . . . . . . . . . . . . . . . . . . . . . . . . 2

Section

I. Introduction . . . . . . . . . . . . . . . . . . . 4

II. Preliminary Material . . . . . . . . . . . . . 6

III. General Results on the Inversion of the Transforms . . . . . . . . . . . . . . . . . . . 14

IV. Longitudinal Waves . . . . . . . . . . . . . . 19

Footnotes . . . . . . . . . . . . . . . . . . . . . . . . 30

Figure 1: Inverse to the Normal Speed Locus . . . . 31

Figure 2: Section of the Ray Conoid for $a_o \neq A_o$ . . 32

Figure 3: Section of the Ray Conoid for $a_o = A_o$ . . 32

# On the Green's Function for Two-Dimensional Magnetohydrodynamic Waves. I

## I. Introduction.

In this paper we shall be concerned with wave motion in a plasma where the plasma is considered as a continuous medium.[1] A particularly simple set of equations describing such a medium is the Lundquist equations,[2] which represent an isentropic fluid with scalar pressure tensor and infinite conductivity, and in which displacement currents have been ignored. We shall examine only small disturbances imposed on a plasma initially with zero flow velocity and constant matter density and magnetic field; that is, we shall employ the Lundquist equations linearized about these quantities. As further restrictions waves depending just on the $x$ and $y$ coordinates will be included and the unperturbed magnetic field is taken in the direction of the x-axis. For such wave motion we shall give the Green's function, or fundamental solution, which represents the wave emanating from a point source. The Alfven and longitudinal waves are automatically decoupled and many simplifications result. That the magnetic field is divergenceless introduces complications. This condition is in fact an initial condition rather than a dynamical equation, and the techniques used are not naturally adapted for the imposition of initial conditions. With an artifice we shall be able to achieve the result of imposing the condition without introducing inconsistencies. We shall treat the Green's function as a distribution[3] and determine it by inverting its Fourier transform obtained from the partial differential equations. The interpretation as a distribution is essential as we shall have to evaluate many singular integrals and the Green's function itself is not integrable considered as an ordinary function.

The author wishes to thank Dr. Clifford Gardner for numerous stimulating discussions and Dr. Harold Grad for his suggestions, many of which have been incorporated into the paper.

## II. Preliminary Material.

In rationalized MKS units we may write the equations for the fundamental solution, or Green's function, for hydromagnetic waves derived from the linearized Lundquist equations as:

$$(1a)\qquad \frac{\partial \rho}{\partial t} + \rho_o \nabla \cdot \vec{u} = S_\rho' \, \delta(\vec{r})\delta(t)$$

$$(1b)\qquad \rho_o \frac{\partial \vec{u}}{\partial t} + a_o^2 \nabla \rho - \frac{1}{\mu} (\nabla \times \vec{B}) \times \vec{B}_o = \vec{S}_u' \, \delta(\vec{r})\delta(t)$$

$$(1c)\qquad \frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{u} \times \vec{B}_o) = \vec{S}_B' \, \delta(\vec{r})\delta(t),$$

where $\rho_o$, $\vec{B}_o$, $a_o$, and $\mu$ are the constant mass density, magnetic field, ordinary sound speed, and magnetic permeability of the undisturbed fluid, and $\rho$, $\vec{B}$, and $\vec{u}$ are the perturbed density, magnetic field and fluid velocity. $\vec{S}_u'$ and $\vec{S}_B'$ are arbitrary constant vectors and $S_\rho'$ is a constant. Although we refer to "the" Green's function, clearly we mean expressions for $\rho$, $\vec{u}$, and $\vec{B}$ satisfying (1). We shall give such expressions, which have the form of sums of various functions multiplied by $S_\rho$ and by components of $\vec{S}_u'$ and $\vec{S}_B'$ for the case under consideration. Temporarily we write the solution in terms of scalars, vectors, and dyadics as:

$$\rho = G_{\rho\rho} \, S_\rho + \vec{G}_{\rho u} \cdot \vec{S}_u' + \vec{G}_{\rho B} \cdot \vec{S}_B'$$

$$\vec{u} = \vec{G}_{u\rho}\ S_\rho + \vec{\vec{G}}_{uu} \cdot\ \vec{S}_u^{\,\prime} + \vec{\vec{G}}_{uB} \cdot\ \vec{S}_B^{\,\prime}$$

$$\vec{B} = \vec{G}_{B\rho}\ S_\rho + \vec{\vec{G}}_{Bu} \cdot\ S_u + \vec{\vec{G}}_{BB} \cdot\ \vec{S}_B^{\,\prime}$$

While the system (1) is perfectly satisfactory mathematically, there is an additional equation that should be included so that the quantities become physically meaningful:

$$\text{(1d)} \qquad \nabla \cdot \vec{B} = 0.$$

However, from (1c), we infer

$$\frac{\partial \nabla \cdot \vec{B}}{\partial t} = \delta(t)\, \nabla \cdot (\delta(\vec{r})\vec{S}_B^{\,\prime}) \neq 0$$

A consideration of the use of Green's functions immediately suggests methods to avoid the contradiction.

If instead of the delta functions on the right hand sides of (1a), (1b), (1c) there were the functions $\tilde{\rho}(\vec{r},t)$, $\tilde{\vec{u}}(\vec{r},t)$, $\tilde{\vec{B}}(\vec{r},t)$, then we might write the solution of the system, which includes the initial value problem as a special case, as:

$$\rho(\vec{r},t) = \int (d\vec{r}\,')(dt')\Big\{G_{\rho\rho}(\vec{r}-\vec{r}\,',t-t')\rho(\vec{r}\,',t')$$

$$+ \vec{G}_{\rho u}(\vec{r}-\vec{r}\,',t-t') \cdot\ \tilde{\vec{u}}(\vec{r}\,',t') + \vec{G}_{\rho B}(\vec{r}-\vec{r}\,',t-t') \cdot \tilde{\vec{B}}(\vec{r}\,',t')\Big\}$$

$$\vec{u}(\vec{r},t) = \int (d\vec{r}\,')(dt')\Big\{\vec{G}_{u\rho}(\vec{r}-\vec{r}\,',t-t')\tilde{\rho}(\vec{r}\,',t')$$

$$+ \overleftrightarrow{G}_{uu}(\vec{r}-\vec{r}\,',t-t')\cdot\tilde{\vec{u}}(\vec{r}\,',t') + \overleftrightarrow{G}_{uB}(\vec{r}-\vec{r}\,',t-t')\cdot\tilde{\vec{B}}(\vec{r}\,',t')\Big\}$$

$$\vec{B}(\vec{r},t) = \int (d\vec{r}\,')(dt')\Big\{\vec{G}_{B\rho}(\vec{r}-\vec{r}\,',t-t')\tilde{\rho}(\vec{r}\,',t')$$

$$+ \overleftrightarrow{G}_{Bu}(\vec{r}-\vec{r}\,',t-t')\cdot\tilde{\vec{u}}(\vec{r}\,',t') + \overleftrightarrow{G}_{BB}(\vec{r}-\vec{r}\,',t-t')\cdot\tilde{\vec{B}}(\vec{r}\,',t')\Big\}.$$

For such a system to be physically realizable $\tilde{\vec{B}}(\vec{r},t)$ must satisfy $\nabla\cdot\tilde{\vec{B}}(\vec{r},t) = 0$. We shall separate $\overleftrightarrow{G}_{BB}$ into two parts, $\overleftrightarrow{G}_{BB} = \overleftrightarrow{G}^{(1)}_{BB} + \overleftrightarrow{G}^{(2)}_{BB}$, so that for any $\tilde{\vec{B}}(\vec{r},t)$ with zero divergence

$$0 = \int (d\vec{r}\,')\overleftrightarrow{G}^{(2)}_{BB}(\vec{r}-\vec{r}\,',t-t')\cdot\tilde{\vec{B}}(\vec{r}\,',t')$$

$$0 = \nabla\cdot\int (d\vec{r}\,')\overleftrightarrow{G}^{(2)}_{BB}(\vec{r}-\vec{r}\,',t-t')\cdot\tilde{\vec{B}}(\vec{r}\,',t').$$

Thus while both $\overleftrightarrow{G}^{(1)}_{BB}$ and $\overleftrightarrow{G}^{(2)}_{BB}$ occur in the Green's function of the original system, for physically acceptable problems we may ignore $\overleftrightarrow{G}^{(2)}_{BB}$. Further, although (1d) is not satisfied with the full $\overleftrightarrow{G}_{BB}$, with only $\overleftrightarrow{G}^{(1)}_{BB}$ (1d) is satisfied. Obviously such a separation is not unique. In order to understand in what sense the separation chosen is a very natural one and to verify its desired properties we consider an alternate form for (1c):

$$\text{(1c')} \qquad \frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{u} \times \vec{B}_o) = \nabla \times (\vec{S}_{A'}\delta(\vec{r}))\delta(t).$$

Now automatically $\partial \nabla \cdot \vec{B}/\partial t = 0$. Again if we solve the inhomogeneous problem with sources $\tilde{\rho}(\vec{r},t)$, $\tilde{\vec{u}}(\vec{r},t)$ and $\tilde{\vec{B}}(\vec{r},t) = \nabla \times \tilde{\vec{A}}(\vec{r},t)$, and with an obvious extension of notation:

$$\tilde{\vec{B}}(\vec{r},t) = \int (d\vec{r}\,')(dt')\Big\{\vec{G}_{B\rho}(\vec{r}-\vec{r}\,',t-t')\tilde{\rho}(\vec{r}\,',t') \\ + \vec{G}_{Bu}(\vec{r}-\vec{r}\,',t-t')\cdot\tilde{\vec{u}}(\vec{r}\,',t') + \vec{G}_{BA}(\vec{r}-\vec{r}\,',t-t')\cdot\tilde{\vec{A}}(\vec{r}\,',t')\Big\}$$

and the other equations are unchanged. After an integration by parts we shall see that the last term in the equation for $\vec{B}$ becomes

$$\int (d\vec{r}\,')(dt')\ \vec{G}^{(1)}_{BB}(\vec{r}-\vec{r}\,',t-t')\cdot\tilde{\vec{B}}(\vec{r}\,',t'),$$

and all the desired properties follow. We need not separate $\vec{G}_{B\rho}$ or $\vec{G}_{Bu}$ into two parts as they contribute an automatically divergence-free magnetic field. Since in the equation for $\vec{u}$ only $\nabla \times \vec{B}$ occurs, only the divergence-free part of $\vec{B}$ is present and we need not make any separation here. We do not replace (1c) by (1c') entirely primarily as a matter of completeness. However, we shall finally give explicit results only for $\vec{G}^{(1)}_{BB}$ and not for $\vec{G}_{BB}$.

All quantities depend only on $x$ and $y$ and the spatial delta functions in (1) now represent $\delta(x)\delta(y)$. The Fourier and Laplace transform of $\rho(\vec{r},t)$ is defined by

$$\rho(\vec{r},t) = \frac{1}{(2\pi)^3}\int (d\vec{k})(d\omega)\rho(\vec{k},\omega)e^{i(\vec{k}\cdot\vec{r}-\omega t)} ,$$

where the $\omega$ integration runs from $-\infty$ to $+\infty$ on a line just above the real axis in the complex $\omega$ plane, and the $\vec{k}$ integration, $\vec{k} = (k_x, k_y, 0)$, $(d\vec{k}) = (dk_x)(dk_y)$, runs over all real vectors $(k_x, k_y)$. We define in a similar manner the transforms of $\vec{u}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$. The symbols $\rho$, $\vec{u}$, $\vec{B}$ represent interchangeably the functions and transforms; context will readily determine which is involved. With the further definitions and substitutions

$$(2)\quad \left\{\begin{array}{l} \vec{B}_o = |B_o|(1,0,0) = |B_o|\vec{b} \\[2ex] A_o^2 = \dfrac{B_o^2}{\mu\rho_o} \text{ , the Alfven speed} \\[2ex] S_\rho' = \rho_o S_\rho \\[2ex] \vec{S}_u' = \rho_o \vec{S}_u \\[2ex] \vec{S}_B' = |B_o|\vec{S}_B \\[2ex] \vec{S}_A' = |B_o|\vec{S}_A \\[2ex] \vec{B} \text{ is replaced by } |B_o|\vec{B} \\[2ex] \rho \text{ is replaced by } \rho_o\rho \end{array}\right.$$

the equations for the transformed variables become

$$(3a) \qquad \omega\rho - \vec{k}\cdot\vec{u} = iS_\rho$$

$$(3b) \qquad \omega\vec{u} - a_o^2\,\vec{k}\rho + A_o^2(\vec{b}\times(\vec{B}\times\vec{k})) = i\vec{S}_u$$

$$(3c) \qquad \omega\vec{B} + \vec{k}\times(\vec{u}\times\vec{b}) = i\vec{S}_B$$

$$(3c') \qquad \omega\vec{B} + \vec{k}\times(\vec{u}\times\vec{b}) = -(k\times S_A).$$

In the case under consideration the Alfven wave does not couple to the rest of the system, thus

$$(4a) \qquad \omega u_z + A_o^2\,k_x B_z = iS_{u_z}$$

$$(4b) \qquad \omega B_z + k_x\,u_z = iS_{B_z},$$

and $u_z$ and $B_z$ do not occur elsewhere in the system. The equations (4) represent an Alfven wave, and a trivial calculation gives:

$$(5a) \qquad u_z(r,t) = \frac{\delta(y)}{2}\Big\{(S_{u_z} - A_oS_{B_z})\delta(x-A_ot) + (S_{u_z} + A_oS_{B_z})\delta(x + A_ot)\Big\}$$

$$(5b) \qquad B_z(r,t) = \frac{\delta(y)}{2A_o}\Big\{(A_oS_{B_z} - S_{u_z})\delta(x-A_ot) + (A_oS_{B_z} + S_{u_z})\delta(x + A_ot)\Big\}.$$

The remaining coupled equations give the longitudinal wave.

With the definition

$$D = \omega^4 - \omega^2 k^2 (a_o^2 + A_o^2) + a_o^2 A_o^2 k^2 k_x^2$$

the solutions of the equations (3a), (3b), (3c) for $\rho$, $u_x$, $u_y$, $B_x$, $B_y$ are:

(6a) $$\rho = \frac{i}{D} \Big\{ \omega(\omega^2 - A_o^2 k^2) S_\rho + \omega^2 (k \cdot S_u) - k^2 k_x A_o^2 S_{u_x} + \omega\, k_y A_o^2 (k_y S_{B_x} - k_x S_{B_y}) \Big\}$$

(6b) $$u_x = \frac{i}{D} \Big\{ \omega(\omega^2 - k^2 A_o^2) S_{u_x} + \omega k_y a_o^2 (k_x S_{u_y} - k_y S_{u_x}) + k_x (\omega^2 - k^2 A_o^2) a_o^2 S_\rho + a_o^2 A_o^2\, k_x k_y (k_y S_{B_x} - k_x S_{B_y}) \Big\}$$

(6c) $$u_y = \frac{i}{D} \Big\{ \omega(\omega^2 - k_x^2 a_o^2) S_{u_y} + k_x k_y \omega a_o^2 S_{u_x} + k_y \omega^2 a_o^2 S_\rho + (\omega^2 - k_x^2 a_o^2) A_o^2 (k_y S_{B_x} - k_x S_{B_y}) \Big\}$$

(6d) $$B_x = \frac{i}{D} \Big\{ \omega(\omega^2 - a_o^2 k^2) S_{B_x} + \omega k_y^2\, a_o^2 S_\rho + \omega^2 k_y S_{u_y} + k_x k_y a_o^2 (k_y S_{u_x} - k_x S_{u_y}) \Big\} + \frac{i}{\omega D} (\vec{k} \cdot \vec{S}_B) k_x A_o^2 (a_o^2 k^2 - \omega^2)$$

(6e) $$B_y = \frac{i}{D} \Big\{ \omega(\omega^2 - a_o^2 k^2) S_{B_y} - \omega k_x k_y a_o^2 S_\rho - \omega^2 k_x S_{u_y} - k_x^2 a_o^2 (k_y S_{u_x} - k_x S_{u_y}) \Big\} + \frac{i}{\omega D} (\vec{k} \cdot \vec{S}_B) k_y A_o^2 (a_o^2 k^2 - \omega^2).$$

In the expressions (6d) and (6e) for $B_x$ and $B_y$ the terms in the second bracket with the leading factor $i\vec{k}\cdot\vec{S}_B$ are the "unphysical" part of the Green's function forming $\vec{G}^{(2)}_{BB}$. The remainder of the terms depending on $S_B$ form $\vec{G}^{(1)}_{BB}$. To justify the separation and the general statements above let us consider equation (3c') instead of (3c), so that we replace $\vec{S}_B$ by $i(\vec{k}\times\vec{S}_A)$. First, all the terms from $\vec{G}^{(2)}_{B}$ vanish identically, so that $\vec{G}^{(2)}_{BB}$ contributes nothing to a physical problem. If we define $J(\vec{r},t)$ as the Fourier transform of $i\omega(\omega^2 - a_o^2k^2)/D$ then exhibiting explicitly the dependence on $S_A$ only, we have

$$B_x = S_{A_z} \frac{\partial}{\partial y} J(\vec{r},t) + \ldots$$

$$B_y = -S_{A_z} \frac{\partial}{\partial x} J(\vec{r},t) + \ldots$$

For the inhomogeneous problem

$$B_x(r,t) = \int (d\vec{r}\,')dt' \frac{\partial}{\partial y} J(\vec{r}-\vec{r}\,',t-t')\tilde{A}_z(\vec{r}\,',t) + \ldots,$$

and by an integration by parts (with no surface terms as $J$ has compact support for all $t$)

$$B_x(\vec{r},t) = \int (d\vec{r}\,')dt' \; J(\vec{r}-\vec{r}\,',t-t') \frac{\partial\tilde{A}_z(\vec{r}\,',t)}{\partial y'} + \ldots$$

$$= \int (d\vec{r}\,')dt' \; J(\vec{r}-\vec{r}\,',t-t')\tilde{B}_x(\vec{r}\,',t') + \ldots$$

and similarly for $B_y(\vec{r},t)$

$$B_y(\vec{r},t) = \int (d\vec{r}\,')dt'\ J(\vec{r}-\vec{r}\,',t-t')B_y(\vec{r}\,',t').$$

However, we note that these are precisely the integrals that occur if we include $\vec{G}_{BB}^{(1)}$ only. Similar remarks apply to $B_z(\vec{r},t)$, but we need not even consider this component as $B_z$ does not appear in $\nabla\cdot B$ since no quantities depend on $z$.

## III. General Results on the Inversion of the Transforms.

Initially we shall consider a general transform defined by:

$$(7) \qquad F(\vec{r},t) = \frac{1}{(2\pi)^3}\int d\omega \int (d\vec{k})e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,\frac{k_x^\alpha k_y^\beta \omega^\gamma}{Q(\omega,\vec{k})}$$

where $Q(\omega,\vec{k})$ is a homogeneous polynomial of degree $m$ which is the characteristic polynomial of a hyperbolic partial differential equation and $\alpha$, $\beta$, $\gamma$ are positive integers whose sum is $m-2$. All the terms in equations (6) are of this form except that $\alpha+\beta+\gamma = m-1$, however by the relation

$$\int d\omega\int(d\vec{k})e^{i(\vec{k}\cdot\vec{r}-\omega t)}(-i\omega\vec{\ell}(\omega,\vec{k}) + i\vec{k}n(\omega,\vec{k}))$$
$$= \frac{\partial}{\partial t}\left(\int d\omega\int(d\vec{k})e^{i(\vec{k}\cdot\vec{r}-\omega t)}\vec{\ell}(\omega,\vec{k})\right)+\vec{\nabla}\left(\int d\omega\int(d\vec{k})e^{i(\vec{k}\cdot\vec{r}-\omega t)}n(\omega,\vec{k})\right)$$

we may express all the transforms in (6) as derivatives of the type considered in (7). We shall use polar coordinates in the $\vec{k}$ plane slightly different from the ordinary ones; the "magnitude" of $\vec{k}$, $k$, will range from $-\infty$ to $+\infty$, and the polar angle $\theta$ from $0$ to $\pi$; the range $-\infty < k \leq 0$, $0 \leq \theta \leq \pi$ corresponding to $0 \leq k \leq \infty$, $\pi \leq \theta \leq 2\pi$ in the ordinary usage. In such a system $(d\vec{k}) = dk_x dk_y = |k| dk\, d\theta = k\, \mathrm{sgn}(k) dk\, d\theta$. The symbol $\hat{k} = \hat{k}(\theta)$ denotes the unit vector in the $\theta$ direction, with the direction $\theta = 0$ to be specified later.

The zeros of the denominator of the integrand in (7), considered as a function of $\omega$, are

$$\omega_j = kv_j(\theta) \qquad j = 1, 2, 3, \ldots, m.$$

The $v_j(\theta)$ are just the speeds of propagation of the various modes in the $\theta$ direction. With the symbol $H(t)$ for the Heaviside function, and after the $\omega$ integration, we have

$$(8) \qquad F(\vec{r},t) = \frac{-iH(t)}{(2\pi)^2} \int_{-\infty}^{\infty} \mathrm{sgn}(k) dk \int_{0}^{\pi} d\theta (\hat{k}_x)^{\alpha} (\hat{k}_y)^{\beta} \left\{ \sum_{(j)} \frac{e^{ik(\hat{k}\cdot\vec{r} - v_j(\theta)t)} (v_j(\theta))^{\gamma}}{Q_{\omega}(v_j(\theta), \hat{k}(\theta))} \right\}$$

where $Q_{\omega}(v_j(\theta), \hat{k}) = \frac{\partial}{\partial\omega} Q(\omega, \hat{k}) \Big|_{\omega = v_j(\theta)}$.

The k integration is meaningful as a distribution. With the integral[4]

$$\int_{-\infty}^{\infty} dk\ \mathrm{sgn}(k) e^{ikz} = \frac{2i}{z}$$

and on identification of the appropriate partial fraction decomposition, we have

$$F(\vec{r},t) = \frac{H(t)}{2\pi^2 t} \int_0^{\pi} d\theta \frac{(\hat{k}_x)^{\alpha} (\hat{k}_y)^{\beta} (\frac{\hat{k}\cdot\vec{r}}{t})^{\gamma}}{Q(\frac{\hat{k}\cdot\vec{r}}{t},\ \hat{k}(\theta))} \quad . \tag{9}$$

For future use we note that the zeros of the denominator of (9) correspond to unit vectors $\hat{k}$ such that

$$\frac{\hat{k}\cdot\vec{r}}{t} = v_j(\hat{k}) \qquad j = 1, 2, 3, \ldots, m. \tag{10}$$

In (8) we have tacitly assumed that the roots of $Q(\omega,\hat{k})$ are simple, however we may derive equation (9) easily even if the roots are multiple or of multiplicity changing with $\theta$ by the appropriate modification of (8).

Since the denominator in (9) has zeros not integrable in the ordinary sense, we must give some interpretation to the integral. One might suppose that the appropriate interpretation is the usual finite part integral considered by various authors, e.g., Lighthill.[5] In that case the integrand in (9) would be extended into the complex $\theta$ plane and the integral would be taken as the average of the two integrals on contours just above and just below the real axis. Such a prescription follows directly from the more standard formulas. However, normally we expect finite part integrals, which are indeed always finite, to represent ordinary functions. Here we expect the answer to be a distribution, which in the ordinary sense need not be finite. It is beyond the scope of this paper to derive rigorously the correct interpretation of such integrals;[6] we shall merely describe and discuss the results. The primary thing to note is that the finite path or contour integral interpretation is indeed valid for strictly hyperbolic equations, i.e., when the wavespeeds, $v_j(\theta)$, $j = 1, 2, \ldots, m$, are distinct for each value of $\theta$, provided none of the roots $v_j(\theta)$ is of the form $\vec{a} \cdot \hat{k}$, with $\vec{a}$ a constant vector. For the non-strictly hyperbolic case, again with no speeds linear in $\hat{k}$, we find that $F(\vec{r},t)$ is given by the contour integral plus some functions discontinuous across certain planes associated with the speeds that are multiple. For hydromagnetic waves the roots are multiple but ultimately the associated complications cancel.

In passing we give some ideas as to how the results are derived rigorously. Provided the multiplicity of the roots is not too great, we may write

$$F(\vec{r},t) = \frac{H(t)}{2\pi^2} \left(\frac{\partial}{\partial x}\right)^{\alpha} \left(\frac{\partial}{\partial y}\right)^{\beta} \left(\frac{\partial}{\partial t}\right)^{\gamma} \int_0^{\pi} d\theta \left\{ \frac{1}{(m-3)!} \sum_{j=1}^{m} \frac{(\hat{k}\cdot\vec{r}-v_j(\theta)t)^{m-3} \log|\hat{k}\cdot\vec{r}-v_j(\theta)t|}{Q_{\omega}(v_j(\theta), \hat{k})} \right\},$$

which follows from plane wave techniques[7] or merely on integration of (9) formally several times in $x$, $y$, and $t$. By taking $F(\vec{r},t)$ as the real part of the integral we should like to remove the absolute value signs from the logarithm, displace the contour of integration and then differentiate, deriving the contour integral representation. All the steps may be justified in the strictly hyperbolic case with wave speeds not linear in $\hat{k}$. If the equations are not strictly hyperbolic, but without linear wave speeds, we find that there are complications with branch cuts of the logarithm, and it is just these jumps that give rise to the discontinuous functions.

## IV. Longitudinal Waves.

For longitudinal waves the polynomial $Q(\omega,\vec{k})$ in (7) is just $\omega^4 - \omega^2(a_o^2 + A_o^2)k^2 + a_o^2 A_o^2 k^2 k_x^2$ so that the wave speeds are given by

$$v_1(\theta) = -v_3(\theta) = \left(\left(\frac{a_o^2+A_o^2}{2}\right) + \frac{1}{2}\left((a_o^2+A_o^2)^2 - 4a_o^2A_o^2(\hat{k}_x)^2\right)^{1/2}\right)^{1/2}$$

(11)

$$v_2(\theta) = -v_4(\theta) = \left(\left(\frac{a_o^2+A_o^2}{2}\right) - \frac{1}{2}\left((a_o^2+A_o^2)^2 - 4a_o^2A_o^2(\hat{k}_x)^2\right)^{1/2}\right)^{1/2}$$

$v_1(\theta)$ and $v_3(\theta)$ being the fast wave and $v_2(\theta)$ and $v_4(\theta)$ the slow wave. For $\hat{k}_x = 0$, $v_2 = v_4 = 0$. Thus the wave speed locus has a double point. Further if $a_o = A_o$ then for $\hat{k}_x = \pm 1$, $v_1 = v_2$ and $v_3 = v_4$ so that another double point may occur. As described in detail elsewhere[6], the double points in the characteristic equation introduce certain planes in x,y,t space where there is irregular behavior of the Green's function not given by the principle value interpretation of (9). For the double root at $\hat{k}_x = 0$, the plane is $y=0$, $(x/t)^2 < a_o^2A_o^2 / a_o^2+A_o^2$, and when $a_o = A_o$, so that $\hat{k}_x = \pm 1$ is another double root, there are planes $x = \pm a_o t$, $(y/x)^2 < 1/4$. However for the combinations of terms that appear in (6), the irregularities associated with these planes cancel identically. In other words, while the planes in question are possible singularity surfaces for systems of equations with this characteristic polynomial, for this particular system they have no effect and we may ignore them and write

$$(12) \qquad F(\vec{r},t) = \frac{H(t)}{4\pi^2 t} \int_{C+C'} d\theta \, \frac{(\hat{k}_x)^\alpha (\hat{k}_y)^\beta (\hat{k}\cdot \frac{r}{t})^\gamma}{Q(\frac{\hat{k}\cdot r}{t}, \hat{k})} ,$$

with the understanding that in (11) equality holds ignoring terms of no effect for the system of equations for hydromagnetic waves. In (12) $C$ is a contour in the complex $\theta$ plane from $0$ to $\pi$ just above the real axis and $C'$ is a contour just below the real axis from $0$ to $\pi$.

If we now choose the $\theta = 0$ direction as the vector $\vec{B}_o$ and call the angle between $\vec{r}$ and $\vec{B}_o$ $\phi$, and finally introduce the auxilliary variable $z = e^{2i(\theta-\phi)}$, we find

$$(13) \qquad F(\vec{r},t) = \frac{2t^3 H(t)}{\pi^2 r^4 i} \int_{\Gamma+\Gamma'} z\,dz \, \frac{(\hat{k}_x(z))^\alpha (\hat{k}_y(z))^\beta (\frac{\hat{k}(z)\cdot r}{t})^\gamma}{C(z)}$$

where

$$(14) \qquad C(z) = z^4 + 4z^3\left(1 - \frac{a_o^2+A_o^2}{r^2}\, t^2 + \frac{a_o^2 A_o^2 t^4}{r^4}\, e^{2i\phi}\right) + z^2\left(6 - \frac{8(a_o^2+A_o^2)}{r^2}\, t^2 + 8a_o^2 A_o^2 \frac{t^4}{r^4}\right) + 4z\left(1 - \frac{a_o^2+A_o^2}{r^2}\, t^2 + \frac{a_o^2 A_o^2 t^4}{r^4}\, e^{-2i\phi}\right) + 1$$

and where $\Gamma$ and $\Gamma'$ are contours just inside and just outside the unit circle taken in a positive sense. Since $\alpha + \beta + \gamma = 2$ the numerator is even in $\hat{k}$ and when multiplied by $z$ and considered as a function of $z$ it is a second degree polynomial in $z$. Thus the value of the integral is $2\pi i$ times the residue at poles inside the unit circle minus the residue at poles outside the unit circle. Some of the geometrical results derived below are well known and follow from general properties of partial differential equations.[9] We rederive them as a matter of completeness and to show how they reappear when employing transform techniques.

If $C(z) = 0$, then also $C(\overline{1/z}) = 0$, so that $\overline{1/z}$ is also a root of $C(z)$ provided $z\bar{z} \neq 1$, or $z$ lies off the unit circle. For various values of the parameters there are three possible situations: four roots of $C(z)$ on the unit circle; two on the unit circle, one inside and one outside; or two inside and two outside. With this information we shall now work with the variable $\hat{k}$ instead of $z$, the relation being real values of $\hat{k}$ correspond to real angles $\theta$ and $z$ on the unit circle, and complex $\hat{k}$ to all other values of $z$.

As was stated (see (10)), the zeros of $C(z)$ occur whenever

$$(15) \qquad (\hat{k}\cdot\vec{\rho})\,\frac{1}{v_j(\hat{k})} = \rho^2, \qquad j = 1, 2, 3, 4$$

where

$$(16) \qquad \vec{\rho}/\rho^2 = \vec{r}/t \ ,$$

i.e., $\vec{\rho}$ is the inverse point in the unit circle to $\vec{r}/t$. To exhibit the conditions geometrically let us consider Figure 1 representing the $\vec{\rho}$ plane. The positive abscissa corresponds to the positive x direction, and the positive ordinate to the positive y direction. In terms of polar coordinates $(r,\psi)$ in the $\vec{\rho}$ plane we plot the curves $(1/v_j(\psi), \psi)$, $j = 1,2,3,4$. Condition (15) implies $(OA)(OB) = (OC)^2$ where AC is perpendicular to OB. Hence OAC and OCB are similar triangles and OCB is a right angle. To determine the real $\hat{k}$ for which (15) holds we find the intersections with the curve $1/v_j$ of the line $\ell$ through $\vec{\rho}$ and perpendicular to the line from $\vec{\rho}$ to the origin. Intersections at infinity must not be neglected, thus lines parallel to the ordinate axis have two intersections at infinity with the $(1/v)$ curve. Clearly any line must intersect the slow curve at least twice, and correspondingly $C(z)$ always has at least two roots on the unit circle.

Except in the neighborhood of those values of the coefficients of $C(z)$ for which there are multiple roots, the roots will be analytic functions of the coefficients and hence of $\vec{r}/t$. Except at these special points the residues of the integral (13) will also be analytic functions of $\vec{r}/t$. Since there is never more than one root of $C(z)$ inside the unit circle or one root outside, it suffices to find the values of $\vec{\rho}$ in Figure 1 which correspond to multiple intersections of th line $\ell(\vec{\rho})$ with the $1/v$ locus. If $\ell$ has a multiple intersection, then $\ell$ is tangent to the curve or the point on the curve is a multiple point. But the multiple points are associated with real angles $\theta$ and hence do not affect the integral. The points in the $\vec{\rho}$ plane at which there is the possibility of non-regular behavior of the integral (13) form the set of points each being the closest to the origin on a tangent to the curve $1/v$. But by definition this curve is just the pedal curve of the original curve. The inverse in the unit circle of the pedal curve, a curve in the ordinary $\vec{r}/t$ space, is well known to be a plane section of the ray cone, the ray cone being a surface carrying discontinuities of the Green's function. We note that for this characteristic polynomial certain other planes may also carry discontinuities not present for this system of equations.

The ray cone is shown in Figure 2 when $a_o \neq A_o$. In regions A and C there are four real intersections of the line $\ell(\vec{\rho})$ and the $1/v$ locus and in region B two. In regions A and C there are four roots of $C(z)$ on the unit circle and in B two. Since roots on the unit circle do not contribute to the integral, in regions A and C the integral vanishes. The vanishing of the integral in region A gives the appropriate range of influence to the space time point $x = y = t = 0$. The line DD' shows the position of the possible discontinuity plane. When $a_o = A_o$ the section of the ray cone is given in Figure 3. Now the lines EE' and FF' may also carry discontinuities. The regions A, B and C have the same general properties in both figures.

Parametric trigonometric representations of the ray cone have been given elsewhere,[9] but we may use its description as the inverse of the pedal curve of the 1/v locus to obtain an algebraic parametric representation. The locus of the 1/v curve is given in rectangluar coordinates as:

$$(17) \qquad Q(\xi,\eta) = 1-(a_o^2 + A_o^2)(\xi^2+ \eta^2)+ a_o^2A_o^2\xi^2(\xi^2+ \eta^2) = 0.$$

The prescription for the ray cone then gives

$$(18a) \qquad \frac{x}{t} \left(\xi \frac{\partial Q(\xi,\eta)}{\partial \xi} + \eta \frac{\partial Q(\xi,\eta)}{\partial \eta}\right) = \frac{\partial Q}{\partial \xi}$$

$$(18b) \qquad \frac{y}{t} \left(\xi \frac{\partial Q(\xi,\eta)}{\partial \xi} + \eta \frac{\partial Q(\xi,\eta)}{\partial \eta}\right) = \frac{\partial Q}{\partial \eta} .$$

We may eliminate $\eta$ between (17) and (18) and obtain

$$(19a) \qquad 0 = a_o^4A_o^4\xi^5 - 2a_o^2A_o^2(a_o^2 + A_o^2)\xi^3 + 2a_o^2A_o^2 \left(\frac{x}{t}\right)\xi^2$$

$$+ ((a_o^2 + A_o^2)^2 - a_o^2A_o^2)\xi -(a_o^2 + A_o^2)\frac{x}{t}$$

$$= \xi((a_o^2A_o^2\xi^2-(a_o^2+A_o^2))^2 - a_o^2A_o^2)- \frac{x}{t}(2a_o^2A_o^2\xi -(a_o^2+A_o^2)) \quad ,$$

$$(19b) \qquad 0 = \left(\frac{y}{t}\right)^2 \left(2a_o^2 A_o^2 \xi^2 - (a_o^2 + A_o^2)\right)^2$$

$$- (a_o^2 + A_o^2 - a_o^2 A_o^2 \xi^2)^3 (1-(a_o^2 + A_o^2)\xi^2 + \xi^4 a_o^2 A_o^2)$$

where $\xi$ is restricted by

$$(20a) \qquad 0 \leq \xi^2 \leq \min\left(\frac{1}{a_o^2}, \frac{1}{A_o^2}\right)$$

or

$$(20b) \qquad \max\left(\frac{1}{a_o^2}, \frac{1}{A_o^2}\right) \leq \xi^2 \leq \frac{a_o^2 + A_o^2}{a_o^2 A_o^2} .$$

As we shall not be concerned with particularly detailed formulae for the Green's function, we shall not try to get the most explicit form of an answer. But if one were concerned with an actual computation, rather than study C(z), a polynomial of fourth degree and with complex coefficients, one could examine a real cubic. In the region where the Green's function is different from zero we may express the roots of C(z) in terms of the real parameters $\lambda$, $\zeta$, and $\eta$ as $e^{i\zeta+\lambda}$, $e^{i\zeta-\lambda}$, $e^{-i(\zeta+\eta)}$, and $e^{-i(\zeta-\eta)}$. From the standard relations between roots and coefficients one can finally get a real cubic with roots $\cos(2\zeta)$, $\cos(i\lambda+\eta)$, and $\cos(i\lambda-\eta)$, and the Green's function may be written in terms of these. We shall adopt a much simpler expedient; $z_i$ and $z_o$ are respectively the roots of

$C(z)$ inside and outside the unit circle, $z_1$ and $z_2$ are the two on the unit circle.

We are now almost ready to give the explicit form of the Green's function. But first we shall make a few comments on the unphysical part of the Green's function, which we shall not include in the formulas. If we define a function $h(\vec{r},t)$ whose Fourier transform is $(\omega^2 - a_o^2 k^2)/D$, then the unphysical part of $\overset{\rightrightarrows}{G}_{BB}$, $\overset{\rightrightarrows}{G}{}^{(2)}_{BB}$ is given by (see (6d), (6e))

$$\overset{\rightrightarrows}{G}{}^{(2)}_{BB} = \nabla\nabla \int_0^t dt' h(\bar{r},t').$$

The function $h(\vec{r},t)$ is just of the type in the physical Green's function, and thus its time integral will have the same general behavior so far as the singularity surfaces as the physical Green's function. It is possible to do the time integral explicitly, but we shall not explore the matter further. In the expressions below we do not include the unphysical parts of B.

We shall express the solutions in terms of the auxilliary real functions:

$$G_1 = \frac{t^3 H(t)}{\pi^2 r^4 i} \int_{\Gamma+\Gamma'} dz \, \frac{z}{C(z)}$$

$$G_2 = \frac{t^3 H(t)}{2\pi^2 r^4 i} \int_{\Gamma+\Gamma'} dz \, \frac{z^2 + 1}{C(z)}$$

$$G_3 = - \frac{t^3 H(t)}{2\pi^2 r^4} \int_{\Gamma+\Gamma'} dz \, \frac{z^2 - 1}{C(z)} ,$$

where the integral is over the previously described contour. Thus we have:

$$(22a)\qquad G_1 = \begin{cases} \dfrac{2t^3 H(t)}{\pi r^4}\left(\dfrac{z_i}{(z_i-z_o)(z_i-z_1)(z_i-z_2)} - \dfrac{z_o}{(z_o-z_i)(z_o-z_1)(z_o-z_2)}\right) & \text{in B} \\ 0 & \text{in A and C.} \end{cases}$$

$$(22b)\qquad G_2 + iG_3 = \begin{cases} \dfrac{2t^3 H(t)}{\pi r^4}\left(\dfrac{z_i^2}{(z_i-z_o)(z_i-z_1)(z_i-z_2)} - \dfrac{z_o^2}{(z_o-z_i)(z_o-z_1)(z_i-z_2)}\right) & \text{in B} \\ 0 & \text{in A and C.} \end{cases}$$

where the regions A, B, C in (22) refer to Figures 2 and 3. By reference to equations (6) and (13) we see that three fundamental forms appear in the physical solution corresponding to $\hat{k}_x^2$, $\hat{k}_y^2$ and $\hat{k}_x\hat{k}_y$ in the numerator of (13). Corresponding to these forms we introduce the Green's functions $G_{xx}$, $G_{xy}$ and $G_{yy}$, so that

$$(23a)\qquad G_{xx} = G_1 + G_2\cos 2\phi - G_3\sin 2\phi$$

$$(23b)\qquad G_{xy} = G_2\sin 2\phi + G_3\cos 2\phi$$

$$(23c)\qquad G_{yy} = G_1 - G_2\cos 2\phi + G_3\sin 2\phi.$$

After some manipulation we have finally:

$$(24a) \quad \rho = (S_{u_x} \frac{\partial}{\partial x} - \frac{\partial}{\partial t} S_\rho)((\frac{x}{t})^2 - A_o^2)G_{xx} + \frac{2xy}{t^2} G_{xy} + \frac{y^2}{t^2} G_{yy})$$

$$+ (A_o^2(S_{B_x} \frac{\partial}{\partial y} - S_{B_y} \frac{\partial}{\partial x}) - S_{u_y} \frac{\partial}{\partial t} - A_o^2 S_\rho \frac{\partial}{\partial y})(\frac{x}{t} G_{xy} + \frac{y}{t} G_{yy})$$

$$- A_o^2 S_{u_x} \frac{\partial}{\partial y} G_{xy}.$$

$$(24b) \quad u_x = (S_\rho \frac{\partial}{\partial x} - S_{u_x} \frac{\partial}{\partial t})\left\{(\frac{x}{t})^2 - A_o^2)G_{xx} + \frac{2xy}{t^2} G_{xy} + \frac{y^2}{t^2} G_{yy}\right\}$$

$$+ (a_o^2 A_o^2(S_{B_x} \frac{\partial}{\partial y} - S_{B_y} \frac{\partial}{\partial x}) - A_o^2 S_\rho \frac{\partial}{\partial y} - a_o^2 S_{u_y} \frac{\partial}{\partial t})G_{xy}$$

$$- (a_o^2 + A_o^2) S_{u_x} \frac{\partial}{\partial y} (\frac{x}{t} G_{xy} + \frac{y}{t} G_{yy}).$$

$$(24c) \quad u_y = \Big(-S_{u_y} \frac{\partial}{\partial t} + A_o^2(S_{B_x} \frac{\partial}{\partial y} - S_{B_y} \frac{\partial}{\partial x})\Big)\Big((\frac{x}{t})^2 - a_o^2)G_{xx}$$

$$+ \frac{2xy}{t^2} G_{xy} + \frac{y^2}{t^2} G_{yy}\Big) + a_o^2(S_{u_x} \frac{\partial}{\partial x} - S_\rho \frac{\partial}{\partial t})(\frac{x}{t} G_{xy} + \frac{y}{t} G_{yy}).$$

$$(24d) \quad B_x = -S_{B_x} \frac{\partial}{\partial t}\Big(((\frac{x}{t})^2 - a_o^2)G_{xx} + \frac{2xy}{t^2} G_{xy} + (\frac{y}{t})^2 G_{yy}\Big)$$

$$+ (a_o^2(S_\rho - S_{B_x}) \frac{\partial}{\partial y} - S_{u_y} \frac{\partial}{\partial t})(\frac{x}{t} G_{xy} + \frac{y}{t} G_{yy})$$

$$+ a_o^2(S_{u_x} \frac{\partial}{\partial y} - S_{u_y} \frac{\partial}{\partial x})G_{xy}$$

$$(24e)\quad B_y = -\left(S_{B_y}\frac{\partial}{\partial t} + S_{u_y}\frac{\partial}{\partial x}\right)\left(\left(\left(\frac{x}{t}\right)^2 - a_o^2\right)G_{xx} + \frac{2xy}{t^2}G_{xy} + \left(\frac{y}{t}\right)^2 G_{yy}\right)$$

$$- a_o^2\left(S_{B_y}\frac{\partial}{\partial y} + S_\rho\frac{\partial}{\partial x}\right)\left(\frac{x}{t}G_{xy} + \frac{y}{t}G_{yy}\right) - a_o^2 S_{u_x}\frac{\partial}{\partial x}G_{xy}.$$

Since the results are so formidible a very qualitative description of the solution may be useful. From (22) we see that all the functions in (24) are different from zero only in region B of Figure 1 and they become infinite as $z_1 \rightarrow z_o$, or as $\vec{r}/t$ approaches the boundary of B, and we may consider the disturbance strongest there. At the points D and D' of Figure 1 the singularity in the Green's function is of a higher degree than elsewhere, so that loosely speaking the disturbance centers on D and D'. When $a_o = A_o$ Figure 2 is relevant and D,D', E,E', F,F' are points of highest singularity.

If we wished to perform the differentiations implicit in (24) we must recall that the expressions are distributions and derivitives of singular distributions are not identical with derivitives of the corresponding functions, e.g.,

$$\frac{\partial}{\partial x}\frac{x}{x^2 + y^2} = \left(\frac{\partial}{\partial x}\right)^2 \log r = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \pi\delta(x)\delta(y).$$

Such singular behavior is expected near D and D'.

FOOTNOTES

1. A preliminary account of the results in this report was presented at the December 1959 meeting of the Plasma Physics Division of the American Physical Society at Monterey, Calif.

2. See, e.g., the paper of H. Grad in The Magnetodynamics of Conducting Fluids, ed. D. Bershader, Stanford University Press (1959), and references cited there.

3. Distributions are used essentially as described in An Introduction to Fourier Analysis and Generalized Functions by M. J. Lighthill, Cambridge University Press (1958). However, there are certain added complications here not present in Lighthill's treatment. In a paper to appear later we study the fundamental solution of a general linear first order homogeneous hyperbolic equation with constant coefficients in two space and one time variables. With classical methods we show the functions given here are derivitives of ordinary functions so that the integral of the fundamental solution times a $C^\infty$ function of compact support is naturally defined by integration by parts.

4. See, e.g., M. J. Lighthill, op. cit.

5. M. J. Lighthill, pp 37-38, op. cit.

6. A preliminary account of these results was presented by the author at the International Conference on Partial Differential Equations and Continuum Mechanics at Madison, Wisconsin June 1959. A paper will appear shortly.

7. F. John, Plane Waves and Spherical Means, Chapter II, Interscience Press (1955).

8. See R. Courant and D. Hilbert, Methods of Mathematical Physics, vol. 2, Interscience Press, to appear. For the general theory as applied to these equations see J. Bazer and O. Fleischman, Phys. Fluids $\underline{2}$, 366 (1959), and H. Grad, op. cit.

9. See J. Bazer and O. Fleischman, op. cit.

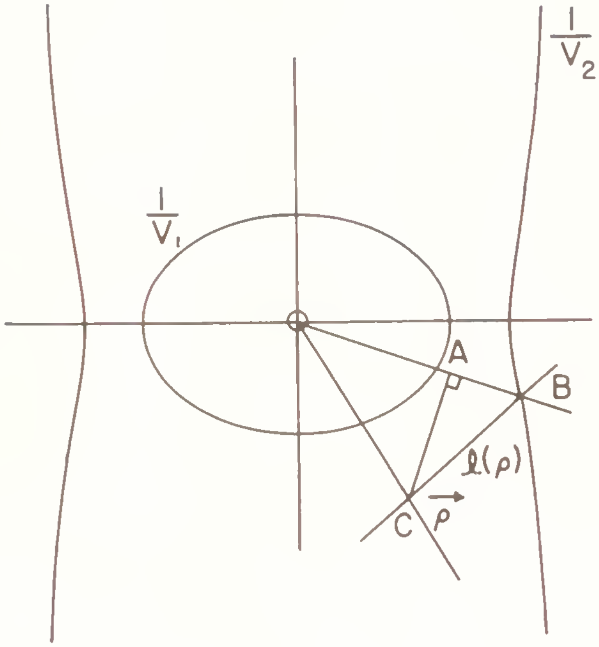

FIG. 1

INVERSE TO THE NORMAL SPEED LOCUS

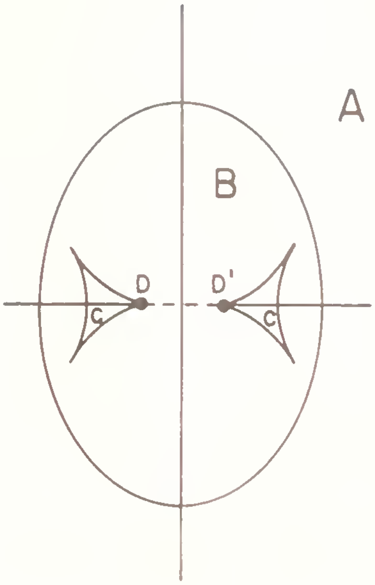

FIG. 2
SECTION OF THE RAY CONOID FOR $a_0 \neq A_0$

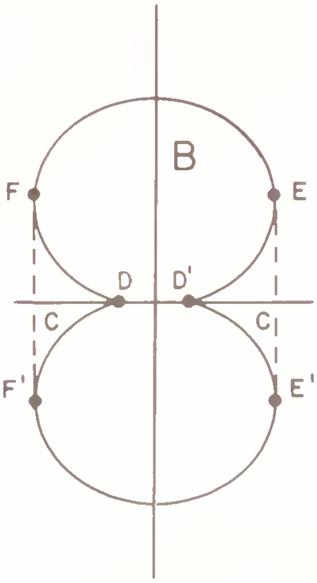

FIG. 3
SECTION OF THE RAY CONOID FOR $a_0 = A_0$

DEC 8 1960

Date Due

MAR 29 '68

PRINTED IN U. S. A.

NYU c.1
NYO-2886
Weitzner
On the Green's function for
two-dimensional ... waves

NYU c. 1
NYO-2886
Weitzner
AUTHOR
On the Green's function...
TITLE

| DATE DUE | BORROWER'S NAME | ROOM NUMBER |
|---|---|---|
| DEC 2 '83 | | |

N. Y. U. Institute of
Mathematical Sciences
25 Waverly Place
New York 3, N. Y.